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Further Studies in Aesthetic Field Theory III: A Bounded Particle in $\Gamma^i_{jk;l} = 0$, $g_{ij;k} = 0$ Theory

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Abstract

In previous papers we showed that certain aesthetic ideas led to a bounded particle. In this paper, we show that a theory based on $\Gamma^i_{jk;l} = 0$, $g_{ij;k} = 0$ with $g = 0$ can also lead to a bounded particle. This theory has the advantage that all tensors constructed from g_{ij}, Γ^i_{jk}, ∂_i are treated in a uniform way. Also, we have sixty-four distinct Γ^i_{jk} appearing. This was not the case in our previous work.

1. *Theory*

After many unsuccessful attempts, we found in previous papers (Muraskin, 1973; Muraskin & Ring, 1973) a bounded particle within aesthetic field theory. In order to achieve this, we assumed that $g = 0$. Because $g = 0$, we cannot raise indices with g_{ij}. Thus, we work in a formalism in which all indices are subscripts. In Muraskin (1972) our field equations were

$$\frac{\partial \Lambda_{ijk}}{\partial x_l} = \Lambda_{mjk}\Lambda_{mil} + \Lambda_{imk}\Lambda_{mjl} + \Lambda_{ijm}\Lambda_{mkl} \tag{1.1}$$

In Muraskin & Ring (1972) our field equations were

$$\frac{\partial e_{\alpha i}}{\partial x_k} = \Lambda_{\ \alpha\beta\gamma} e_{\beta i} e_{\gamma k} \tag{1.2}$$

But such field equations do not lead to all tensors being treated in a uniform way, so far as their change is concerned. This could be viewed as a disturbing feature from the point of view of aesthetics. Thus, a problem is whether we can formulate a $g = 0$ theory which treats tensors, and higher derivatives, in a uniform fashion.

There was also another more technical problem that we encountered. We found that after an $e_{\alpha i}$ transformation at the origin, we did not get sixty-four different Λ_{ijk}.

In this paper, we shall discuss a $g = 0$ theory in which tensor functions of g_{ij}, Γ^i_{jk} and ∂_i are treated in a uniform way and in which sixty-four Γ^i_{jk} are distinct.

The key point in our argument is this. Since $g = 0$, we cannot use g_{ij} to raise and lower indices. However, by introducing a set of basis vector variables, we can define contravariant indices using the reciprocal (dual) basis vector field. In order for this to be possible, it is necessary to alter the boundary conditions suggested in our previous papers. There we required the basis vector fields to go to zero at infinity. But this would imply that we cannot have a reciprocal basis vector field that is finite everywhere. Thus, instead, we shall postulate a set of basis vectors $e^\alpha{}_i$ which obey $e^\alpha{}_i \to \delta^\alpha_i$ at infinity. Then, we can introduce the reciprocal field $e_\alpha{}^j$ satisfying

$$e^\alpha{}_i \, e_\alpha{}^j = \delta^\alpha_\beta \qquad e^\alpha{}_i \, e_\beta{}^i = \delta^\alpha_\beta \tag{1.3}$$

at all points. Next, we require that the change of $e^\alpha{}_i$ be given by ($\bar{\Gamma}^\alpha{}_{\beta k}$ is assumed present so that $R^i{}_{jkl} \neq 0$ may be considered)

$$de^\alpha{}_i = \Gamma^j_{ik} \, e^\alpha{}_j \, dx^k - \bar{\Gamma}^\alpha{}_{\beta k} \, e^\beta{}_i \, dx^k \tag{1.4}$$

Then from (1.3) we get

$$de_\alpha{}^j = -\Gamma^j_{ik} \, e_\alpha{}^i \, dx^k + \bar{\Gamma}^\beta{}_{\alpha k} \, e_\beta{}^j \, dx^k \tag{1.5}$$

We require that Γ^j_{ik} determine the change of subscripted i, j indices according to the first term of (1.4) and superscripted i, j indices according to the first term of (1.5). This gives, for the change of Γ^i_{jk},

$$\frac{\partial \Gamma^i_{jk}}{\partial x^l} = \Gamma^i_{mk} \, \Gamma^m_{jl} + \Gamma^i_{jm} \, \Gamma^m_{kl} - \Gamma^m_{jk} \, \Gamma^i_{ml} \tag{1.6}$$

This is the same equation introduced in our earliest work (Muraskin, 1970). This equation has the property that all tensor products of g_{ij}, Γ^i_{jk} and ∂_i are treated in a uniform way so far as their change is concerned. g_{ij} is introduced by means of

$$g_{ij} = e^\alpha{}_i \, e^\beta{}_j \, g_{\alpha\beta}(x) \tag{1.7}$$

at the origin

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.8}$$

Thus, we have $g = 0$. The change of g_{ij} is again the same as in our earlier work (Muraskin, 1970)

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^t_{ik} \, g_{tj} + \Gamma^t_{jk} \, g_{it}. \tag{1.9}$$

We cannot introduce g^{ij} satisfying $g^{ij} g_{jk} = \delta^i_k$ any more, nor is there a $g^{\alpha\beta}$ satisfying $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$. We shall not introduce any $g^{\alpha\beta}$ into the theory.

Since we have not introduced g^{ij}, we do not have such objects as $g^{ij}\Gamma^t_{im}\Gamma^m_{jt}$. Thus, the restrictions that these objects must be zero in order to satisfy the boundary conditions $\Gamma^i_{jk} \to 0$ at infinity does not appear in the present theory.

We may ask if our $g = 0$ theory can satisfy the following:

(1) Integrability.
(2) $\Gamma^i_{jk} \to 0$ and $g_{ij} \to 0$ at infinity.
(3) A particle appearing at an arbitrary origin.

We have worked with the following data. The $\Gamma^\alpha_{\beta\gamma}$ chosen to be nonzero are

$$\begin{aligned} &\Gamma^1_{10} = \Gamma^2_{20} = \Gamma^3_{30} = \Gamma^0_{00} = \Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = A \\ &\Gamma^0_{11} = \Gamma^0_{22} = \Gamma^0_{33} = -B \\ &\Gamma^2_{13} = \Gamma^3_{21} = \Gamma^1_{32} = -\Gamma^1_{23} = -\Gamma^3_{12} = -\Gamma^2_{31} = C \end{aligned} \tag{1.10}$$

This data obeys $R^i{}_{jkl} \neq 0$ in general, but still satisfies the integrability equations. When $A = B = C$, we get $R^i{}_{jkl} = 0$.

For our purposes, we have considered the following choice of parameters as representative of $R^i{}_{jkl} \neq 0$ theory

$$A = 1, \qquad B = 1, \qquad C = 0{\cdot}7 \tag{1.11}$$

At the origin we have

$$\Gamma^i_{jk} = e_\alpha{}^i e^\beta{}_j e^\gamma{}_k \Gamma^\alpha_{\beta\gamma} \tag{1.12}$$

with $e^\alpha{}_i$ chosen to be

$$\begin{array}{llll} e^1{}_1 = 0{\cdot}7 & e^1{}_2 = 0{\cdot}62 & e^1{}_3 = 0{\cdot}46 & e^1{}_0 = 2{\cdot}4 \\ e^2{}_1 = -0{\cdot}12 & e^2{}_2 = -0{\cdot}08 & e^2{}_3 = -0{\cdot}14 & e^2{}_0 = 0{\cdot}082 \\ e^3{}_1 = -0{\cdot}015 & e^3{}_2 = -0.097 & e^3{}_3 = -0{\cdot}0111 & e^3{}_0 = 0{\cdot}092 \\ & & & e^0{}_0 = 2{\cdot}0 \end{array} \tag{1.13}$$

$e^0{}_1$, $e^0{}_2$, $e^0{}_3$ were then calculated according to the procedures in Muraskin (1971) using $g_{\alpha\beta} = (1, 1, 1, 0)$. With this set of data we get a maximum in g_{00}.

Equation (1.12) implies that $\Gamma^\alpha_{\beta\gamma}$ must be a function of x in order that $\Gamma^i_{jk} \to 0$ when $e^\alpha{}_i \to \delta^\alpha_i$ at infinity.

We have found that our computer results seem to suggest $g_{ij} \to 0$ at infinity. This means that $g_{\alpha\beta}$ should be a function of x, just as $\Gamma^\alpha_{\beta\gamma}$ is.

We have not yet found a $\Gamma^\alpha{}_{\beta k}$ set of data at the origin to confirm that $e^\alpha{}_i \to \delta^\alpha_i$ at infinity. We shall suppose, without proof, that such a set does exist.

The data (1.10) which appears at an arbitrary origin point is unchanged by any three-dimensional spatial rotation.†

We now get that all sixty-four Γ^i_{jk} are different, and thus the problem of repeats which was present in our Λ_{ijk} and $e_{\alpha i}$ work is gone.

† This fact has also been confirmed for various angles using the computer.

2. *Computer Results*

We have obtained the following:

(a) As a result of running down the coordinate axes, g_{00} was found to be bounded by zero in all cases. g_{00} was also found to be bounded by zero in all other selected directions that we tried.

(b) In addition to the maximum in g_{00} at the origin, we have found a minimum nearby.

(c) Along the coordinate axes we have found that all components of the field tend to zero after we have gone far from the origin.

(d) g_{00} as a function of x, y, z can be characterized as follows. Along $+x, +y, +z$, g_{00} monotonically approaches zero. Along $-x, -y, -z$, g_{00} decreases, then increases, and then decreases again tending towards zero.

These results are similar to what we have obtained in previous papers using somewhat different field equations, but with data at the origin that is basically of the same character as the data used here.†

If, in (1.11), we choose $A = B = C = 1$, we get an example of $R^i{}_{jkl} = 0$ theory. We can get the same results for Γ^i_{jk} and g_{ij} at each point by using instead of (1.6) and (1.9), the equations

$$\frac{\partial e^\alpha{}_i}{\partial x^k} = \Gamma^\alpha_{\beta\gamma} e^\beta{}_i e^\gamma{}_k \tag{2.1}$$

with

$$\begin{aligned} \Gamma^i_{jk} &= e_\alpha{}^i e^\beta{}_j e^\gamma{}_k \Gamma^\alpha_{\beta\gamma} \\ g_{ij} &= e^\alpha{}_i e^\beta{}_j g_{\alpha\beta} \end{aligned} \tag{2.2}$$

with $\Gamma^\alpha_{\beta\gamma}$ and $g_{\alpha\beta}$ constant. Integrability of $e^\alpha{}_i$ leads to $R^i{}_{jkl} = 0$. From (2.1) and (2.2), we get (1.6) and (1.9). We have confirmed that (2.1) and (2.2) lead to the same answers for Γ^i_{jk} and g_{ij} as (1.6) and (1.9) to computer accuracy.‡ However, using (2.1) we find $e^\alpha{}_i \to 0$ far down the axis, and thus we do not get $e^\alpha{}_i \to \delta^\alpha_i$. Thus, we can no longer introduce contravariant indices at all points using these basis vectors. We nevertheless made some computer runs in the $R^i{}_{jkl} = 0$ theory. Our results are the same as (a), (b), (c), (d) given previously. In fact, we may go one step beyond this. We have found, from all the computer runs we have made, that the values of g_{ij} are not sensitive

† In all cases the data was unchanged by three-dimensional spatial rotations or was form-invariant under four-dimensional rotations. However, the data in this latter case [see equation (5.4) of the preceding paper, p. 90] can be obtained from a four-dimensional orthogonal transformation on data that is unchanged by three-dimensional spatial rotations. Thus, in either situation our data is consistent with an underlying structure that is invariant under three-dimensional spatial rotations.

‡ Although we get the same answers when $g_{\alpha\beta}$, $\Gamma^\alpha_{\beta\gamma}$ are constant we have not argued that we cannot get the same answers when $g_{\alpha\beta}$, $\Gamma^\alpha_{\beta\gamma}$ are not constant.

to the choice of C in (1.11) provided we use the same $e^{\alpha}{}_{i}$ as in (1.13).† Thus, we get the same values of g_{ij} at each point for the $R^{i}{}_{jkl} = 0$ and $R^{i}{}_{jkl} \neq 0$ data even though Γ^{i}_{jk} are different for the two cases. If $C = 1$, we have formally the same data as in our $e_{\alpha i}$ work based on equation (1.2) as well. We have also confirmed that we get the same values of g_{ij} at all points there (that we considered), as compared with the present case.

3. *Conclusions*

In this paper, we have shown that a theory based on $\Gamma^{i}_{jkl} = 0$, $g_{ij;k} = 0$ and $g = 0$ can lead to a bounded particle (as inferred from axes runs and selected runs off the axes). The bounded particle appears similar to the kind of particle we have been getting previously when we used different equations. Our present theory has the advantage that all tensors constructed from Γ^{i}_{jk}, g_{ij}, ∂_{i} are treated in a uniform way so far as their change is concerned. Also, we have sixty-four distinct Γ^{i}_{jk} appearing.

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References

Muraskin, M. (1970). *Annals of Physics*, **59**, 27.
Muraskin, M. (1971). *International Journal of Theoretical Physics*, Vol. 4, No. 1, p. 49.
Muraskin, M. (1973). *International Journal of Theoretical Physics*, Vol. 7, No. 3, p. 213.
Muraskin, M. and Ring, B. (1973). *International Journal of Theoretical Physics*, Vol. 8, No. 2, p. 85.

† Analytically, we have shown from the field equations that $\left(\frac{\partial g_{ij}}{\partial x^k}\right)$ and $\left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^m}\right)$ and $e^{0}{}_{1}$, $e^{0}{}_{2}$, $e^{0}{}_{3}$ are independent of C. In order to complete the proof that g_{ij} is independent of C at all points we would have to show that all derivatives of g_{ij} are independent of C.